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VARIATIONAL PRINCIPLES FOR BOUNDARY VALUE AND INITIAL-BOUNDARY VALUE PROBLEMS IN CONTINUUM MECHANICS

RANBIR S. SANDHU

The Ohio State University, Columbus, Ohio 43210

and

KARL S. PISTER

University of California, Berkeley, California 94720

Abstract—A procedure for setting up variational principles for a class of linear coupled field problems in continuum mechanics is presented. Some generalizations of the principle and their relationship with existing variational theorems are examined. Alternative schemes useful for direct methods of solution are discussed. Examples of typical application are included.

1. INTRODUCTION

USE of variational methods for obtaining approximate solutions to field problems and for establishing constitutive equations in continuum mechanics is well established [1–7]. Several alternative formulations have been described for specific classes of problems [8–11]; however, the common underlying basis of these various works has received inadequate attention. Sandhu and Pister [12] recently presented a variational principle applicable to a wide class of linear coupled field problems. Their approach is based on a generalization of the basic variational problem of Mikhlin [13] to the case of functions of several dependent variables. For initial boundary value problems a bilinear form suggested by the work of Gurtin [8, 9] is used.

In the present paper we observe that symmetry properties of the matrix of operators for the class of problems considered lead to several extended variational principles such that the unique intersection of the sets of approximate solutions associated with these alternative formulations is the problem solution. Direct numerical methods can be used in conjunction with the several variational principles to determine sets of approximate solutions. We also note that linear symmetric operators can be decomposed by introduction of additional dependent variables, corresponding to "splitting" of a problem in the sense of Synge [3]. Through this device a linear symmetric problem can be replaced by a symmetric linear coupled problem and multiple variational principles established.

In Section 2, we outline a procedure for setting up alternative variational principles. Some generalizations are introduced, extending the range of application to a class of nonlinear problems. Section 3 illustrates the procedures developed, using the problem of elastostatics to demonstrate the relationship between the concepts introduced herein and some existing formulations. Finally, Section 4 presents additional examples using decomposition of symmetric operators and appropriate transformation of the field equations to generate the required linear symmetric coupled formulation in each case.

The extended variational principles and illustrative examples discussed, used in conjunction with direct numerical methods such as finite difference or finite element, provide a basis for constructing approximate solutions to a wide class of problems in continuum mechanics.

2. A VARIATIONAL PRINCIPLE

For a positive, linear, bounded operator A defined on a dense set $M \subseteq V$, a Hilbert space over the field F, the equation

$$Au = f \tag{1}$$

has a unique solution and if the solution exists, the functional

$$\Omega(u) = \langle u, Au \rangle - 2\langle u, f \rangle \tag{2}$$

assumes its minimum value for this solution. Here we restrict our discussion to a space of real valued functions. The symbol $\langle a, b \rangle$ denotes the inner product of $a, b \in V$. The operator A is positive if

$$\langle u, Au \rangle > 0; \qquad u \in V, \qquad u \neq 0 \tag{3}$$

and

$$\langle u, Av \rangle = \langle Au, v \rangle; \qquad u, v \in V. \tag{4}$$

An alternative statement of the principle involves a variation $\Delta\Omega$ of the functional Ω, defined as:

$$\Delta\Omega = \Omega(v) - \Omega(u). \tag{5}$$

Then, the value of Ω at u is minimum if the variation $\Delta\Omega$ is positive for all $v \in V$. We refer the reader to Mikhlin [13] for a detailed discussion of the foregoing principle. We observe here that other definitions for variation of the functional can be employed. Komkov [14] used Fréchet derivatives of Ω. The first Fréchet derivative of Ω with respect to u is defined as $Z \in V$ such that

$$\langle Z, v \rangle = \lim_{\lambda \to 0} \frac{\Omega(u + \lambda v) - \Omega(u)}{\lambda} \tag{6}$$

where λ is real, and $v \in V$. In this manner the problem is reduced to the classical form requiring the first derivative of a quantity to vanish at a point where it acquires an extremal or stationary value. We shall write the first Fréchet derivative of Ω with respect to u as

$$Z = \frac{\partial \Omega}{\partial u}. \tag{7}$$

Vanishing of the variation of a functional will imply vanishing of the first Fréchet derivative of Ω with respect to its argument. Second Fréchet derivatives are introduced [14] to define extremum or stationary properties of the functional. For quadratic functionals defined by

(2), the second Fréchet derivative is the operator A. Thus the extremal characteristics in the two approaches, using different definitions for the variation of the functional, are seen to be equivalent.

We now state an extension for several dependent variables: V is defined as the direct sum $V_1 \oplus V_2 \oplus \ldots \oplus V_n$, and an element u of V is an n-tuple $(u_1, u_2, \ldots, u_n)$, with $u_i \in V_i$ for $i = 1, 2, \ldots, n$. An inner product on V is now defined as:

$$\langle u, v\rangle = \langle u_1, v_1\rangle + \langle u_2, v_2\rangle + \ldots + \langle u_n, v_n\rangle \text{ (no sum on } n).\dagger \tag{8}$$

The operator A is defined as a two-dimensional array of linear bounded operators, with $n \times n$ elements, such that an element A_{ij} maps a dense set $M_{ij} \subseteq V_j$ into V_i. Equation (4) defining symmetry of the operator A will now mean

$$\langle u_i, A_{ij}u_j\rangle = \langle u_j, A_{ji}u_i\rangle \text{ (no sums).} \tag{9}$$

For diagonal elements of the array, (9) is identical to (4). The operator equation (1) is now a set of linear equations for the problem:

$$A_{ij}u_j = f_i; \quad i, j = 1, 2, \ldots, n. \tag{10}$$

For each $i = 1, 2, \ldots, n$ at least one of the elements $A_{ij}, j = 1, 2, \ldots, n$ and for each $j = 1, 2, \ldots, n$ at least one of the elements $A_{ij}, i = 1, 2, \ldots, n$ must be distinct from the zero operator. We require that the elements of the array A_{ij} satisfy, for homogeneous boundary conditions‡

$$A_{ij}u_j = 0 \text{ if, and only if, } u = 0, i, j = 1, 2, \ldots, n. \tag{11}$$

Diagonal elements of the array A_{ij} will be designated as positive, zero or negative depending upon $\langle u_k, A_{kk}u_k\rangle$ (no sums) being $>$, $=$, or $<$ zero. The governing functional will achieve a minimum/stationary/maximum value with respect to variation in each element of the n-tuple satisfying (10) depending upon the corresponding diagonal element of the array A_{ij} being positive, zero or negative. In the present discussion, however, we will not discuss extremal properties of governing functionals; only the vanishing of the variation is of direct interest.

(a) *Case of two dependent variables*

For two dependent variables, under homogeneous boundary conditions, equations (10) are:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}. \tag{12}$$

† Summation convention on repeated indices is implied in the sequel, except where "no sum" is specifically indicated.

‡ Homogeneous boundary conditions are assumed throughout except where specifically indicated. The treatment is easily extended to non-homogeneous conditions. An appropriate modification of the definition of symmetry of the operator A in (4) is required. This will result in additional terms being added to the functional (2).

Using (8) as the definition of an inner product, the governing functional (2) is

$$\Omega = \langle u_1, A_{11}u_1\rangle + \langle u_1, A_{12}u_2\rangle + \langle u_2, A_{21}u_1\rangle + \langle u_2, A_{22}u_2\rangle - 2\langle u_1, f_1\rangle - 2\langle u_2, f_2\rangle. \tag{13}$$

Taking this form as the basic functional, we shall establish alternative variational principles and introduce certain useful generalizations.

(b) *Consequences of symmetry of A_{ij}*

Symmetry of the operator matrix implies

$$\langle u_1, A_{12}u_2\rangle = \langle u_2, A_{21}u_1\rangle. \tag{14}$$

Using (14) to eliminate the term containing A_{21} in (13) we obtain

$$\Omega_1 = \langle u_1, A_{11}u_1\rangle + 2\langle u_1, A_{12}u_2\rangle + \langle u_2, A_{22}u_2\rangle - 2\langle u_1, f_1\rangle - 2\langle u_2, f_2\rangle. \tag{15}$$

Alternatively, elimination of the term containing A_{12} leads to

$$\Omega_2 = \langle u_1, A_{11}u_1\rangle + 2\langle u_2, A_{21}u_1\rangle + \langle u_2, A_{22}u_2\rangle - 2\langle u_1, f_1\rangle - 2\langle u_2, f_2\rangle. \tag{16}$$

If A_{12}, A_{21} are zero operators or scalar multipliers, Ω_1 and Ω_2 are identical. However, in general, (15) and (16) define two different functionals. We note that Ω is defined for $u_1 \in M_{11} \cap M_{21}, u_2 \in M_{12} \cap M_{22}$ whereas Ω_1 is defined for $u_1 \in M_{11}, u_2 \in M_{12} \cap M_{22}$ and Ω_2 is defined for $u_1 \in M_{11} \cap M_{21}, u_2 \in M_{22}$. Consequently, if the complement of $M_{11} \cap M_{21}$ in M_{11} denoted by $C(M_{11} \cap M_{21})$ is nonempty, the domain of definition of Ω_1, includes and is greater than that of Ω insomuch as certain $u_1 \notin M_{21}$ may be admissible. Similarly, the domain of definition of Ω_2 may include certain $u_2 \notin M_{12}$. An approximate solution using either (15) or (16) as the governing functional may not coincide with the solution obtained from (13). An important aspect of the alternative formulations is that the intersection of the domain of Ω_1 and Ω_2 is the domain of Ω. In an approximate solution scheme, if the choice of admissible u_1 and u_2 is restricted to the domain of Ω, the three formulations (13), (15) and (16) are identical. However, if extended domains are admitted for Ω_1 and Ω_2, the first of the equations (12) is not meaningful for Ω_2 and the second of the equations (12) is not meaningful for Ω_1. The implications of this result for direct solution methods will be found in the following sections.

(c) *A generalization to nonlinear diagonal elements of A_{ij}*

Depending upon A_{11}, A_{22} being positive or negative, we introduce positive or negative valued functionals, $\bar{U}$, $\bar{V}$ such that

$$\bar{U} = \int_F U\,\mathrm{d}F = \tfrac{1}{2}\langle u_1, A_{11}u_1\rangle \tag{17}$$

$$\bar{V} = \int_F V\,\mathrm{d}F = \tfrac{1}{2}\langle u_2, A_{22}u_2\rangle \tag{18}$$

and define $W = U + V$, so that $\bar{W} = \bar{U} + \bar{V}$. (19)

Substituting (17)–(19) into the basic functional (13) we obtain

$$\Omega_3 = \langle u_1, A_{12}u_2\rangle + \langle u_2, A_{21}u_1\rangle + 2\int_F W\,\mathrm{d}F - 2\langle u_1, f_1\rangle - 2\langle u_2, f_2\rangle. \tag{20}$$

Vanishing of variation of Ω_3 yields the equations

$$A_{12}u_2 + \frac{\partial \overline{W}}{\partial u_1} = f_1$$

and

$$A_{21}u_1 + \frac{\partial \overline{W}}{\partial u_2} = f_2 \tag{21}$$

where $\partial \overline{W}/\partial u_1$, $\partial \overline{W}/\partial u_2$ are Fréchet derivatives of $\overline{W}$ with respect to u_1 and u_2 respectively. Clearly, equations (12) are equivalent to (21) if $\overline{W}$ has the properties

$$\begin{aligned} A_{11}u_1 &= \frac{\partial \overline{W}}{\partial u_1} \\ A_{22}u_2 &= \frac{\partial \overline{W}}{\partial u_2}. \end{aligned} \tag{22}$$

Linearity of operators A_{11}, A_{22} is not necessary for (21) to be meaningful. Thus, (20) can be used as a governing functional for problems involving nonlinear diagonal elements in the operator matrix so long as $\overline{W}$ can be defined consistent with (22). This generalization forms the basis of variational equations for a class of nonlinear problems. We notice that $\overline{U}$ and $\overline{V}$ satisfy the following relations:

$$\begin{aligned} \frac{\partial \overline{U}}{\partial u_1} &= A_{11}u_1, & \frac{\partial \overline{U}}{\partial u_2} &= 0 \\ \frac{\partial \overline{V}}{\partial u_2} &= A_{22}u_2, & \frac{\partial \overline{V}}{\partial u_1} &= 0. \end{aligned} \tag{23}$$

Komkov [14] and Reissner [5] use equations (21), referred to as the canonical equations of elasticity, as the starting point in their formulations of the problem of elastostatics of continua.

(d) *A formulation using the inverse of* A_{22}

In some problems a quantity u_3 is introduced such that

$$u_3 = A_{22}u_2. \tag{24}$$

For example, in the elastostatics of continua, u_3 would be interpreted as the measure of deformation or the strain tensor associated with the stress tensor u_2. The inverse relationship, if it exists, would express the stress tensor as a function of the strain tensor. Equations (12) along with (24) can be written as a set of three equations

$$\begin{bmatrix} A_{11} & A_{12} & 0 \\ 0 & A_{22} & -1 \\ A_{21} & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ 0 \\ f_2 \end{Bmatrix}. \tag{25}$$

The matrix of operators in (25) is not symmetric. However, if A_{22} possesses an inverse A_{22}^{-1} such that

$$u_2 = A_{22}^{-1} u_3 \tag{26}$$

we can write

$$\begin{bmatrix} -A_{11} & 0 & -A_{12} \\ 0 & A_{22}^{-1} & -1 \\ -A_{21} & -1 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_3 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} -f_1 \\ 0 \\ -f_2 \end{Bmatrix} \tag{27}$$

which is symmetric in the sense of (9). It is apparent that if A_{11}, A_{22} are positive, the governing functional is maximized, is minimized and remains stationary with respect to variation in u_1, u_3 and u_2 respectively, by the solution satisfying the equation (12) along with (26). The governing functional for equations (27) is

$$\begin{aligned} \Omega_4 = & -\langle u_1, A_{11}u_1\rangle - \langle u_1, A_{12}u_2\rangle + \langle u_3, A_{22}^{-1}u_3\rangle \\ & -2\langle u_3, u_2\rangle - \langle u_2, A_{21}u_1\rangle + 2\langle u_1, f_1\rangle + 2\langle u_2, f_2\rangle. \end{aligned} \tag{28}$$

We define a functional $\bar{X}$ such that

$$\bar{X} = \int_F X \, \mathrm{d}F = \tfrac{1}{2}\langle u_3, A_{22}^{-1}u_3\rangle \tag{29}$$

Then (28) is

$$\begin{aligned} \Omega_5 = & -\langle u_1, A_{11}u_1\rangle - \langle u_1, A_{12}u_2\rangle + 2\int_F X \, \mathrm{d}F - 2\langle u_3, u_2\rangle \\ & -\langle u_2, A_{21}u_1\rangle + 2\langle u_1, f_1\rangle + 2\langle u_2, f_2\rangle. \end{aligned} \tag{30}$$

Vanishing of variation of Ω_5 leads to the field equations

$$\begin{aligned} A_{11}u_1 + A_{12}u_2 &= f_1 \\ A_{21}u_1 + u_3 &= f_2 \end{aligned} \tag{31}$$

provided $\bar{X}$ has the following properties

$$\begin{aligned} \frac{\partial \bar{X}}{\partial u_1} &= 0 \\ \frac{\partial \bar{X}}{\partial u_2} &= 0 \\ \frac{\partial \bar{X}}{\partial u_3} &= u_2. \end{aligned} \tag{32}$$

Clearly, if $\bar{X}$ can be defined with the foregoing properties, existence of A_{22}^{-1} and linearity of A_{22} are not essential. This represents a significant generalization. Thus it is possible to state the problem in the form of equations (30)–(32) without requiring it to be derivable from (13) as the starting point. Herrmann's [7] variational theorem for incompressible materials can be regarded as an example for the case of a linear solid.

If we assume that u_1, u_3 are restricted so that the second of the equations (31) is identically satisfied, we obtain from (30)

$$\Omega_6 = -\langle u_1, A_{11}u_1\rangle + 2\int_F X \,\mathrm{d}F + 2\langle u_1, f_1\rangle \tag{33}$$

A variational principle based on Ω_6 would replace the first of equations (31). $\bar{X}$ would now be defined by (29) and $(31)_2$ in case A_{22}^{-1} exists. This scheme corresponds to the so called "potential energy" formulation in the case of elastostatics of continua.

Defining

$$\int_F V\,\mathrm{d}F = \langle u_2, u_3\rangle - \int_F X\,\mathrm{d}F \tag{34}$$

the governing functional (30) becomes, with change in sign

$$\Omega_7 = \langle u_1, A_{11}u_1\rangle + \langle u_1, A_{12}u_2\rangle + 2\int_F V\,\mathrm{d}F$$
$$+\langle u_2, A_{21}u_1\rangle - 2\langle u_1, f_1\rangle - 2\langle u_2, f_2\rangle. \tag{35}$$

We notice that Ω_7 does not contain u_3 and is essentially equivalent to (20). Vanishing of the variation of Ω_7 yields the equations

$$\begin{aligned} A_{11}u_1 + A_{12}u_2 &= f_1 \\ A_{21}u_1 + A_{22}u_2 &= f_2 \end{aligned} \tag{36}$$

if $\bar{V}$ has the properties:

$$\begin{aligned} \frac{\partial \bar{V}}{\partial u_1} &= 0 \\ \frac{\partial \bar{V}}{\partial u_2} &= A_{22}u_2. \end{aligned} \tag{37}$$

Again A_{22} may be nonlinear and need not possess an inverse. It is only necessary to set up $\bar{V}$ satisfying (37) to make Ω_7 meaningful. In case A_{11} is the zero operator and u_2 is restricted to a class identically satisfying the first of equations (36), the governing functional reduces to

$$\Omega_8 = 2\int_F V\,\mathrm{d}F - 2\langle u_2, f_2\rangle \tag{38}$$

Ω_8 is the so called "complementary energy" functional in the case of elastostatics of continua.

We observe here that (35)–(37) follow directly from (20)–(23) without using the definition (34).

3. THE ELASTOSTATICS PROBLEM

We now proceed to apply the principles stated in the preceding section to the problem of elastostatics. This problem is discussed in detail to illustrate the procedures for setting up alternative variational formulations. Some additional applications are mentioned in Section 4.

The field equations for isothermal quasi-static deformation of anisotropic, linear elastic solids, assuming no initial stresses and strains, are:†

$$\sigma_{ki,k}+f_i = 0 \tag{39}$$

$$u_{(i,j)} = C_{ijkl}\sigma_{kl} \tag{40}$$

on an open bounded connected set F contained in the three-dimensional Euclidean space E, along with the boundary conditions

$$\sigma_{ji}n_j = \hat{t}_i \text{ on } S_1 \tag{41}$$

$$u_i = \hat{u}_i \text{ on } S_2 \tag{42}$$

where S_1, S_2 are complementary subsets of S, the boundary of F. Here u_i, σ_{ij} are defined on $\bar{F}$, the closure of F, and denote respectively the components of the displacement vector, and the symmetric Cauchy stress tensor. C_{ijkl} are components of the isothermal compliance tensor and f_i are components of the body force vector. Rectangular cartesian coordinates are implied, and parentheses around a pair of subscripts denote the symmetric part of a tensor. $\hat{u}_i$, $\hat{t}_i$ are components of the specified displacements and tractions, n_j are components of a unit vector normal to S_1 and S is the union of a finite number of non-interesting closed regular surfaces.

Admitting (u_i, σ_{ij}) as the 9-tuple of dependent variables, components of vectors and tensors being regarded as ordered subsets in an n-tuple, we write the field equations (39) and (40) as

$$\begin{bmatrix} 0 & \frac{1}{2}\left(\delta_{ik}\frac{\partial}{\partial_j}+\delta_{jk}\frac{\partial}{\partial_i}\right) \\ -\frac{1}{2}\left(\delta_{ki}\frac{\partial}{\partial_l}+\delta_{li}\frac{\partial}{\partial_k}\right) & C_{klij} \end{bmatrix} \begin{Bmatrix} u_i \\ \sigma_{ij} \end{Bmatrix} = \begin{Bmatrix} -f_k \\ 0 \end{Bmatrix} \tag{43}$$

where δ_{ij} is Kronecker's delta. Corresponding to the basic functional (13) allowing for nonhomogeneous boundary conditions, we have

$$J_1 = \int_F (u_i\sigma_{ij,j}-u_{i,j}\sigma_{ij}+\sigma_{kl}C_{klij}\sigma_{ij}+2u_if_i)\,\mathrm{d}F - \int_{S_1} u_i(\sigma_{ij}n_j-2\hat{t}_i)\,\mathrm{d}S+\int_{S_2}(u_i-2\hat{u}_i)\sigma_{ij}n_j\,\mathrm{d}S. \tag{44}$$

Using symmetry properties of the operator matrix, we obtain the two alternative formulations corresponding to (15) and (16). Introducing a complementary energy function V to replace the term $\frac{1}{2}\sigma_{kl}C_{klij}\sigma_{ij}$, we obtain two alternative forms of Reissner's general variational equation [1, 5].

$$J_2 = \int_F (V+u_i\sigma_{ij,j}+u_if_i)\,\mathrm{d}F-\int_{S_1} u_i(\sigma_{ij}n_j-\hat{t}_i)\,\mathrm{d}S-\int_{S_2}\hat{u}_i\sigma_{ij}n_j\,\mathrm{d}S \tag{45}$$

$$J_3 = \int_F (V-u_{i,j}\sigma_{ij}+u_if_i)\,\mathrm{d}F+\int_{S_1} u_i\hat{t}_i\,\mathrm{d}S+\int_{S_2}(u_i-\hat{u}_i)\sigma_{ij}n_j\,\mathrm{d}S. \tag{46}$$

† For the three-dimensional problem discussed, the subscripts take on the range of values 1–3.

If we restrict σ_{ij} such that the equilibrium equations (39) and stress boundary conditions (41) are satisfied, J_2 reduces to the functional for minimum complementary energy.

$$J_4 = \int_F V\,\mathrm{d}F - \int_{S_2} \hat{u}_i\sigma_{ij}n_j\,\mathrm{d}S. \tag{47}$$

Alternatively, if u_i is required to satisfy the constitutive equations (40) and displacement boundary conditions (42), J_3 yields a generalization of Castigliano's second theorem using the complementary energy [15].

$$J_5 = \int_F (V - u_i f_i)\,\mathrm{d}F - \int_{S_1} u_i\hat{t}_i\,\mathrm{d}S. \tag{48}$$

In the absence of body forces, for homogeneous boundary conditions, the functional corresponding to (20) is

$$J_6 = \langle u_1, A_{12}u_2\rangle + \langle u_2, A_{21}u_1\rangle + 2\int_F V\,\mathrm{d}F \tag{49}$$

$$= \int_F (u_i\sigma_{ij,j} - u_{i,j}\sigma_{ij} + 2V)\,\mathrm{d}F. \tag{50}$$

The field equations arising from stationary properties of (49) are

$$A_{12}\sigma_{ij} = \sigma_{ij,j} = -\frac{\partial \overline{V}}{\partial u_i} = 0 \tag{51}$$

$$A_{21}u_i = -u_{(i,j)} = -\frac{\partial \overline{V}}{\partial \sigma_{ij}}. \tag{52}$$

Equations (51) and (52) are Hamilton's canonical equations stated by Komkov [14]. J_6 is the Lagrangian and $-2\overline{V}$ is the Hamiltonian in this context.

A parallel path can be followed, in which a strain energy function is used rather than complementary energy. Assuming that the compliance tensor possesses an inverse denoted by components E_{ijkl} (the isothermal elasticity tensor), corresponding to (27) we now write, introducing a 15-tuple $(u_i, e_{ij}, \sigma_{ij})$,

$$\begin{bmatrix} 0 & 0 & -\frac{1}{2}\left(\delta_{ik}\frac{\partial}{\partial_j} + \delta_{jk}\frac{\partial}{\partial_i}\right) \\ 0 & E_{ijkl} & -1 \\ \frac{1}{2}\left(\delta_{ki}\frac{\partial}{\partial_l} + \delta_{li}\frac{\partial}{\partial_k}\right) & -1 & 0 \end{bmatrix} \begin{Bmatrix} u_i \\ e_{kl} \\ \sigma_{ij} \end{Bmatrix} = \begin{Bmatrix} f_k \\ 0 \\ 0 \end{Bmatrix} \tag{53}$$

where $e_{ij} = C_{ijkl}\sigma_{kl}$ are defined as components of the strain tensor.

Allowing for the boundary conditions (41), (42) and using a non-negative strain energy density function X to replace the quantity $\frac{1}{2}e_{ij}E_{ijkl}e_{kl}$ we obtain the functional

$$J_7 = \int_F (2X - u_i\sigma_{ij,j} - 2e_{ij}\sigma_{ij} + u_{i,j}\sigma_{ij} - 2u_i f_i)\,\mathrm{d}F$$

$$+ \int_{S_1} u_i(\sigma_{ij}n_j - 2\hat{t}_i)\,\mathrm{d}S - \int_{S_2} (u_i - 2\hat{u}_i)\sigma_{ij}n_j\,\mathrm{d}S \tag{54}$$

The functional $\bar{X} = \int_F X \, \mathrm{d}F$ has to satisfy the properties (32). Using symmetry properties (9) of the operator matrix in (53) leads to the following alternative formulations of the Hu–Washizu [1] principle of generalized potential energy,

$$J_8 = \int_F (X - e_{ij}\sigma_{ij} - u_i\sigma_{ij,j} - u_i f_i) \, \mathrm{d}F + \int_{S_1} u_i(\sigma_{ij}n_j - \hat{t}_i) \, \mathrm{d}S + \int_{S_2} \hat{u}_i\sigma_{ij}n_j \, \mathrm{d}S \tag{55}$$

and

$$J_9 = \int_F (X - e_{ij}\sigma_{ij} + u_{i,j}\sigma_{ij} - u_i f_i) \, \mathrm{d}F - \int_{S_1} u_i\hat{t}_i \, \mathrm{d}S - \int_{S_2} (u_i - \hat{u}_i)\sigma_{ij}n_j \, \mathrm{d}S. \tag{56}$$

In (56) restricting u_i, e_{ij} to satisfy the third of equations (55), the strain–displacement equations, we obtain the functional for "the total energy theory" [10] of Tonti,

$$J_{10} = \int_F (X - u_i f_i) \, \mathrm{d}F - \int_{S_1} u_i\hat{t}_i \, \mathrm{d}S - \int_{S_2} (u_i - \hat{u}_i)\sigma_{ij}n_j \, \mathrm{d}S. \tag{57}$$

Further restricting u_i to satisfy displacement boundary conditions (42) the functional for Green's variational principle [5] is obtained.

$$J_{11} = \int_F (X - u_i f_i) \, \mathrm{d}F - \int_{S_1} u_i\hat{t}_i \, \mathrm{d}S. \tag{58}$$

For linear elasticity, retaining $\frac{1}{2}e_{ij}E_{ijkl}e_{kl}$, instead of (58) we obtain the functional for potential energy theory [1],

$$J_{12} = \int_F (\tfrac{1}{2}e_{ij}E_{ijkl}e_{kl} - u_i f_i) \, \mathrm{d}F - \int_{S_1} u_i\hat{t}_i \, \mathrm{d}S. \tag{59}$$

Requiring σ_{ij} to satisfy the equations (39) and (41), J_8 leads to

$$J_{13} = \int_F (X - e_{ij}\sigma_{ij}) \, \mathrm{d}F + \int_{S_2} \hat{u}_i\sigma_{ij}n_j \, \mathrm{d}S. \tag{60}$$

Defining

$$X = \tfrac{1}{2}e_{ij}E_{ijkl}e_{kl} = e_{ij}\sigma_{ij} - \tfrac{1}{2}\sigma_{ij}C_{ijkl}\sigma_{kl} = e_{ij}\sigma_{ij} - V \tag{61}$$

we see that J_7, J_8, J_9, J_{13} are equivalent to J_1, J_2, J_3 and J_4 respectively. Indeed in literature we find that the latter are often derived on the basis of the Legendre transformation defined by (61).

In the application of direct methods for construction of approximate solutions, the formulations customarily used are: (1) minimum potential energy (59), where the admissible u_i are such that the constitutive equations (40) and displacement boundary conditions (42) are identically satisfied and the variational principle is equivalent to equation (39), the stress equations of equilibrium; and (2) minimum complementary energy (47), where σ_{ij} are such that the stress equilibrium equation (39) and stress boundary condition (41) are identically satisfied and the variational principle is equivalent to the constitutive equation (40). The sets of solutions obtained from the two schemes are disjoint and constitute bounds [4] to the exact solution, in the sense of the value of the functional $\langle u_2, A_{22}u_2 \rangle$. Equations (45) and (46) state so-called mixed formulations where u_i, σ_{ij} are admitted as the state variables and the variational principle is equivalent to both the field equation (43) along with the boundary conditions.

It is apparent that the several variational principles developed above follow from the symmetry of the operator matrix in (43) and the existence of an inverse for C_{ijkl}, along with generalizations accompanying the introduction of scalar functions V and X. Evidently then, problems of this class in other branches of continuum mechanics should permit similar treatment and it should be possible to apply to these problems the direct methods successfully used in linear elastostatics, see for example [12].

4. SOME APPLICATIONS

For application of the procedures outlined in the previous sections to typical problems in continuum mechanics we shall state the relevant field equations in the linear symmetric form (10), without going over the entire ground of setting up the different variational principles in each case. These can be developed by procedures entirely analogous to those for the elastostatics problem discussed in Section 3. It is unnecessary to list the entire range of applications. The illustrative examples presented here are merely representative. The first example involves splitting of a problem by introduction of an additional dependent variable. Self-adjoint operators can be treated in this fashion. The second example uses integral operations to achieve symmetry and the third example presents a more involved procedure. Throughout, homogeneous boundary conditions are assumed in the interest of brevity. Inclusion of non-homogeneous boundary conditions is a straightforward extension.

Example 1

Saint Venant's torsion problem. The field equation for torsion of prismatic bars can be written in terms of the warping function $\phi(x_1, x_2)$ as [3]

$$\nabla^2\phi = \phi_{,\lambda\lambda} = 0 \tag{62}$$

where Greek subscripts take on the range of values 1, 2. The boundary conditions are

$$\phi_{,\alpha}n_\alpha = e_{\alpha\beta}x_\beta n_\alpha \tag{63}$$

where $e_{\alpha\beta}$ is the permutation symbol.

The harmonic operator in (62) is positive in the sense of (3) and (4). It is, therefore, possible to set up a variational principle using the functional (2) for a single dependent variable along with appropriate boundary terms. However, to obtain disjoint sets of approximate solutions using alternative extensions to the domain of definition of the governing functional it is necessary to introduce additional variables and to set up a coupled symmetric problem. We notice that the Laplacian operator can be regarded as a composition of a divergence operator and a gradient operator. Thus,

$$\nabla^2\phi = \nabla \cdot \nabla\phi. \tag{64}$$

Introducing a vector valued variable

$$\sigma = \mu[\nabla\phi + \psi] \tag{65}$$

where ψ is a solenoidal vector, and μ is a real number, we have

$$\nabla \cdot \sigma = 0 \tag{66}$$

We notice that operator $\nabla\cdot$ is the adjoint of $-\nabla$ and (64) and (66) can be written as

$$\begin{bmatrix} 0 & \nabla\cdot \\ -\nabla & \dfrac{1}{\mu} \end{bmatrix} \begin{Bmatrix} \phi \\ \sigma \end{Bmatrix} = \begin{Bmatrix} 0 \\ \psi \end{Bmatrix} \tag{67}$$

Equation (67) is of the same form as (12) and lends itself to all the alternative variational formulations discussed in Section 2. We recall here that equation (65) expresses shearing stresses in terms of the warping function and (66) is the equation of equilibrium.

Example 2

The piezoelectric elastodynamic problem [16]. The field equations for motion of piezoelectric, elastic solids can be written:†

$$\begin{aligned} g*\sigma_{ij,j}+b_i &= \rho u_i \\ \sigma_{ij} &= E_{ijkl}e_{kl}-e_{kij}E_k \\ e_{ij} &= u_{(i,j)} \\ D_n &= e_{nkl}e_{kl}+P_{nk}E_k \\ E_i &= -\phi_{,i} \\ D_{i,i} &= \bar{Q} \end{aligned} \tag{68}$$

where u_i, e_{ij}, σ_{ij}, E_{ijkl} have the same meaning as in Section 3; E_i, D_n are components of the electric field and electric displacement vectors respectively; e_{ijk}, P_{ij} are components of the piezoelectric stress and permittivity tensor, respectively; ϕ is the electric potential and $\bar{Q}$ is the body charge density. In addition $g(t) = t$, ρ is the mass density of the solid and b_i are components of the modified body force vector defined by

$$b_i(\mathbf{x}, t) = g*f_i(\mathbf{x}, t)+\rho(\mathbf{x})(tv_i(\mathbf{x})+d_i(\mathbf{x})) \tag{69}$$

where

$$u_i(\mathbf{x}, 0) = d_i(\mathbf{x}) \text{ and } \dot{u}_i(\mathbf{x}, 0) = v_i(\mathbf{x}) \tag{70}$$

are the initial conditions.

The notation $*$ denotes a convolution product

$$u*v = \int_0^t u(\mathbf{x}, t-\tau)v(\mathbf{x}, \tau)\,\mathrm{d}\tau \tag{71}$$

or the Stieltjes integral [9]

$$u*\mathrm{d}v = \int_{\tau=-\infty}^{t} u(\mathbf{x}, t-\tau)\,\mathrm{d}v(\mathbf{x}, \tau) \tag{72}$$

provided the integrals are meaningful. For initial boundary value problems we use a bilinear form (instead of the inner product) defined on the linear vector space V as the starting point for setting up the governing functional [12]. The bilinear function maps $V\times V$ into S, a

† We have written the stress equations of motion (68), following Gurtin [8], in a form suitable to incorporate initial conditions into the functional (2).

linear vector space, and has the properties

$$
\begin{aligned}
\langle u, v\rangle &= \langle v, u\rangle \\
\langle u, 0\rangle &= 0 \\
\langle u, v\rangle &= 0 \Rightarrow v = 0 \text{ or } u = 0 \text{ on } S.
\end{aligned} \tag{73}
$$

The Gurtin product defined as

$$\langle u, v\rangle = \int_F (u*v)\,\mathrm{d}F \tag{74}$$

is a bilinear map with the properties (73). The variation is defined after Gurtin [8, 9]

$$\Delta\Omega = \frac{\mathrm{d}}{\mathrm{d}\lambda} \quad \Omega\{u+\lambda v\}\bigg|_{\lambda=0} \tag{75}$$

where λ is real and $u+\lambda v \in M$, the domain of definition of the operator A for $u, v \in V$. We say that the variation of the functional vanishes at u over M if, and only if, $\Delta\Omega$ exists and equals zero for all $v \in V$ such that $u+\lambda v \in M$. We require in this case that $Au = 0$ if, and only if, $u = 0$ over S and the symmetric property (4) is satisfied.

Equations (68) in the 22-tuple $(u_i, e_{ij}, \sigma_{ij}, E_i, D_i, \phi)$ are not symmetric in the sense of (9). They can be put in symmetric form by operating on $(68)_{2-6}$ with the operator $g*(\,)$. The resulting system of equations can be written as $Au = f$, where

$$A = \begin{bmatrix}
\rho\delta_{ik} & 0 & -g*\frac{1}{2}\left(\delta_{ik}\frac{\partial}{\partial_j}+\delta_{jk}\frac{\partial}{\partial_i}\right) & 0 & 0 & 0 \\
0 & g*E_{ijkl} & -g* & -e_{kij}g* & 0 & 0 \\
g*\frac{1}{2}\left(\delta_{ki}\frac{\partial}{\partial_l}+\delta_{li}\frac{\partial}{\partial_k}\right) & -g* & 0 & 0 & 0 & 0 \\
0 & -e_{nkl}g* & 0 & -P_{nk}g* & g* & 0 \\
0 & 0 & 0 & g^* & 0 & g*\frac{\partial}{\partial n} \\
0 & 0 & 0 & 0 & -g*\frac{\partial}{\partial n} & 0
\end{bmatrix}$$

$$u = \begin{Bmatrix} u_i \\ e_{kl} \\ \sigma_{ij} \\ E_k \\ D_n \\ \phi \end{Bmatrix} \quad \text{and } f = \begin{Bmatrix} b_k \\ 0 \\ 0 \\ 0 \\ 0 \\ -g*\bar{Q} \end{Bmatrix} \tag{76}$$

Combining the basic functional (2) with the bilinear map (73), along with an appropriate definition of variation operator (75), permits the direct construction of a governing functional for the piezoelectric elastodynamic problem. Alternative formulations, using symmetry properties (9) of the operator matrix are possible along the lines indicated in Section 2 along with extensions to nonlinear forms. The example has been written for homogeneous boundary conditions and includes mechanical and electrostatic body forces. Extension to nonhomogeneous boundary conditions is direct. A particular case of the functional associated with (76) corresponding to a "potential energy" formulation has been presented for use in a direct calculation method employing finite elements [17].

Example 3

Fluid–structure interaction (dam and reservoir) [18]. The equations of linear elastic coupled isothermal motion of an idealized dam and reservoir system have been written for a finite element discretization, in terms of values of displacements at nodal points of the dam and fluid pressure at nodal points in the reservoir as [18]:

$$Ku+M\ddot{u}-L\pi = f \tag{77}$$

$$S\ddot{u}+H\pi+P\ddot{\pi} = 0 \tag{78}$$

where matrix L is transpose of S, and K, M, H, P are positive definite matrices. u is the displacement vector and π the fluid pressure vector. These equations are not symmetric. To ensure symmetry, Zienkiewicz has reported [18] the following device suggested by Irons. From (78), assuming H^{-1} exists,

$$\pi = -H^{-1}[S\ddot{u}+P\ddot{\pi}]. \tag{79}$$

Substituting (79) in (77)

$$Ku+M\ddot{u}+LH^{-1}S\ddot{u}+LH^{-1}P\ddot{\pi} = f \tag{80}$$

and operating on (79) with P and rearranging terms,

$$PH^{-1}S\ddot{u}+PH^{-1}P\ddot{\pi}+P\pi = 0. \tag{81}$$

Equations (80), (81) are symmetric and can be written as

$$\begin{bmatrix} K+[M+LH^{-1}S]\dfrac{\partial^2}{\partial t^2} & LH^{-1}P\dfrac{\partial^2}{\partial t^2} \\ PH^{-1}S\dfrac{\partial^2}{\partial t^2} & PH^{-1}P\dfrac{\partial^2}{\partial t^2}+P \end{bmatrix} \begin{Bmatrix} u \\ \pi \end{Bmatrix} = \begin{Bmatrix} f \\ 0 \end{Bmatrix}. \tag{82}$$

Equation (82) is of the same form as (12) and it is possible to set up alternative variational formulations discussed in Section 2. To allow for the time dependence, one can utilize either the frequency domain analysis separating the modes or a time domain analysis after suitable transformations to incorporate the initial conditions explicitly into the problem [8]. With the latter approach the Gurtin product (74) is used as the starting point for writing the functionals along with vanishing of variation defined by (75).

5. DISCUSSION

A method is presented for setting up alternative variational principles for a class of linear coupled problems in continuum mechanics. The procedure is quite general and problems involving linear symmetric operators can be decomposed to yield a set of coupled equations of this class. Generalization to problems with nonlinear operators along the diagonal is direct. It is shown that complementary energy type principles can be derived without introducing strain energy type formulations and a Legendre transformation. Using symmetric properties, generalizations to nonlinear diagonal operators and specializations to satisfy some of the field equations and/or boundary conditions identically, leads to several possible direct methods of approximate solution of boundary value and initial boundary value problems. As the use of symmetry properties introduces disjoint extensions to the domain of the functional, certain bounds to the correct solution are obtained.

The examples discuss application to elastostatics, piezoelectricity and hydroelasticity. Applications of coupled problems in elastodynamics, viscoelasticity and thermoelasticity have been mentioned elsewhere [12]. Complex problems of the class discussed also arise in electro and magneto-elasticity, electro and thermo-osmosis as well as in the continuum theory of mixtures [21]. In fact any attempt towards a complete listing of the areas of continuum mechanics in which the principles presented can be usefully employed would be futile and hence is not pursued here. For details of computational procedures involved in use of variational principles in conjunction with direct methods, we refer the reader to literature on the subject [4, 6, 17, 19, 20].

REFERENCES

[1] K. Washizu, *Variational Methods in Elasticity and Plasticity*. Pergamon (1968).
[2] Y. C. Fung, *Foundations of Solid Mechanics*. Prentice-Hall (1965).
[3] J. L. Synge, *The Hyper circle in Mathematical Physics*. Cambridge University Press (1957).
[4] B. M. Fraejis de Veubeke, Upper and Lower Bounds in Matrix Structural Analysis, *Matrix Methods of Structural Analysis*, edited by B. M. F. de Veubeke. Pergamon (1964).
[5] E. Reissner, On Variational Principles in Elasticity, *Proc. Symp. Appl. Math.*, Vol. 8, pp. 1–6. McGraw-Hill (1958).
[6] T. H. H. Pian and P. Tong, Basis of finite element methods for solid continua. *Int. J. Numerical Meth. Engng* **1**, 3–21 (1969).
[7] L. R. Herrmann, Elasticity equations for incompressible and nearly incompressible materials by a variational theorem. *AIAA Jnl.* **3**, **10**, 1896–1900 (1964).
[8] M. E. Gurtin, Variational principles for linear elastodynamics. *Archs ration Mech. Analysis* **16**, 234–250 (1964).
[9] M. E. Gurtin, Variational principles in the linear theory of viscoelasticity. *Archs ration. Mech. Analysis* **13**, 179–191 (1963).
[10] E. Tonti, Variational principles in elastostatics. *Meccanica* **2**, 201–208 (1967).
[11] R. E. Nickell and J. L. Sackman, Variational principles for linear coupled thermoelasticity. *Q. appl. Math.* **XXVI**, 11–26 (1968).
[12] R. S. Sandhu and K. S. Pister, A variational principle for linear coupled field problems. *Int. J. Engng Sci.* in press.
[13] S. G. Mikhlin, *The Problem of the Minimum of a Quadratic Functional*. Holden-Day (1965).
[14] V. Komkov, On a variational formulation of problems in the classical continuum mechanics of solids. *Int. J. Engng Sci.* **6**, 695–720 (1968).
[15] J. P. Den Hartog, *Advanced Strength of Materials*. McGraw-Hill (1952).
[16] H. F. Tiersten, *Linear Piezoelectric Plate Vibrations*. Plenum Press (1969).
[17] H. Allik and T. J. R. Hughes, Finite element method for piezoelectric vibration. *Int. J. Numerical Meth. Engng.* in press.
[18] O. C. Zienkiewicz, Discussion of proceeding paper 6297 by A. K. Chopra, Earthquake behavior of reservoir–dam systems. *J. Engng Mech. Div. Am. Soc. civ. Engrs* **95**, EM3, 801–803 (1969).
[19] R. E. Nickell and J. L. Sackman, Approximate solutions in linear, coupled thermoelasticity. *J. appl. Mech.* **35**, **2**, 255 (1968).

[20] O. C. ZIENKIEWICZ and Y. K. CHEUNG, *The Finite Element Method in Structural and Continuum Mechanics*. McGraw-Hill (1967).
[21] R. S. SANDHU, Fluid Flow in Saturated Porous Elastic Media, Ph.D. Dissertation, University of California at Berkeley, California (1968).

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Абстракт—Дается процесс для бостановления вариационных принципов, касающихся класса задач для линейных сопряженных полей в механике сплошной среды. Исследуются некоторые обобщения принципа и их зависимости с существующими вариационными теоремами. Обсуждаются альтернативные схемы, пригодные для простых методов решения. Даются примеры типичных применеий.